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2d SCFTs from M2-branes

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Abstract: We consider the low-energy limit of the two-dimensional theory on k M2-branes suspended between a straight M5-brane and a curved M5-brane. We argue that it is described by an $\mathcal{N}$=(2, 2) supersymmetric gauge theory with no matter fields but with a non-trivial twisted superpotential, and also by an $\mathcal{N}$=(2, 2) supersymmetric Landau-Ginzburg model, such that the (twisted) superpotentials are determined by the shape of the M5-branes. We find particular cases realize Kazama-Suzuki models. Evidence is provided by the study of ground states, chiral rings, BPS spectra and S^2 partition functions of the systems.

Keywords: M2-branes, two-dimensional superconformal field theory

Contents

1 Introduction and Summary

Consideration of multiple p-branes suspended between other branes is an effective way to study the dynamics of p-dimensional supersymmetric field theories [1, 2]. In this paper, we analyze the dynamics of multiple M2-branes suspended between two M5-branes in the following setup [3]. We use the coordinates $x^{0,\cdots,10}$, with a compactified x^{10} direction. Let us introduce complex combinations $v = x^4 + ix^5$ and $t = \exp(x^7 + ix^{10})$. Then we have, as summarized in Figure 1,

- an M5-brane extending along $x^{0,1,2,3}$ and on the complex one-dimensional curve $t = t(v)$, at a fixed position $(x^6, x^8, x^9) = (L, 0, 0)$,
- an M5-brane (which we call the M5′-brane) extending along directions $x^{0,1,8,9}$ and v, at a fixed potition $(x^2, x^3, x^6, t) = (0, 0, 0, 1)$, and
- k M2-branes extending along $x^{0,1}$ and suspended between the M5 and the M5′ along the x^6 direction.

	x^0	x^1	x^2	x^3	v	x^6	t	x^8	x^9
M5	$-$	$-$	$-$	$-$	$-$	L	$t(v)$	0	0
M5′	$-$	$-$	0	0	$-$	0	1	$-$	$-$
M2	$-$	$-$	0	0	σ	$-$	1	0	0

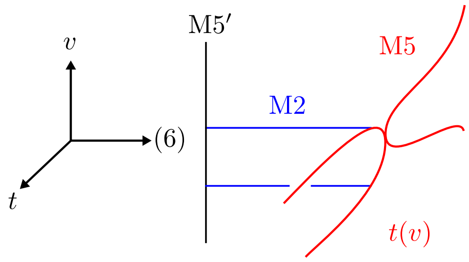


Figure 1: Configuration of branes. $v = x^4 + ix^5$ and $t = \exp(x^7 + ix^{10})$.

We are interested in the infrared limit of the theory on k M2-branes.

If we reduce the theory along the x^{10} direction, we have a system of k D2-branes suspended between one NS5-brane at $x^6 = 0$ and some configuration of branes at $x^6 = L$. This gives a 3d U(k) gauge theory formulated on an interval with some boundary conditions at the two ends, which reduces to a 2d theory with $\mathcal{N}$=(2, 2) supersymmetry at distances longer than the length L of the interval. We first assume that all solutions to the equation $t(v) = 1$ are non-degenerate. That is, if $\{v_i\}_{i\in I}$ denotes the set of solutions, then $t'(v_i) \neq 0$ for each $i \in I$. Then, from the M-theory description, the supersymmetric vacua are described by k M2-branes, separated along v directions, each at a fixed value of v being one of $\{v_i\}_{i\in I}$. The s-rule [1, 3] forbids that more than one M2-brane have the same value of v. This vacuum structure would arize if the low energy theory is the theory on the Coulomb branch with the twisted superpotential

$$\mathcal{W}_{\text{eff}} = \operatorname{tr} P(\Sigma_T), \tag{1.1}$$

for the fieldstrength superfield $\Sigma_T = \operatorname{diag}(\Sigma_1, \ldots, \Sigma_k)$ for the maximal torus $T \cong \mathrm{U}(1)^k$, where the holomorphic function $P(v)$ is given by

$$\exp(P'(v)) = t(v). \tag{1.2}$$

Indeed, the vacuum equation is $t(\sigma_a) = 1$ for $a = 1, \ldots, k$ and we expect that no supersymmetric vacuum is supported at the solutions with $\sigma_a = \sigma_b$ for $a \neq b$. Also, permutations of σ_a's are gauge symmetry.

When $t(v)$ is a rational function, the M5 reduces to a number of D4-branes ending on an NS5-brane, and the 2d theory can be interpreted as a U(k) gauge theory with a number of fundamental and antifundamental chiral multiplets, possibly with twisted masses [3]. There are only finitely many solutions to $t(v) = 1$ and hence the number of supersymmetric vacua is finite. When $t(v)$ is such that $P'(v)$ in (1.2) is a polynomial, the 2d theory has a different type of interpretation: It is the U(k) gauge theory without matter field and with the tree level twisted superpotential

$$\mathcal{W} = \operatorname{tr} P(\Sigma) + \pi i(k + 1) \operatorname{tr} \Sigma, \tag{1.3}$$

where Σ is now the fieldstrength for the full $\mathrm{U}(k)$ vector mutiplet. The second term is the theta term with $\theta = \pi(k+1)$. It is non-trivial if and only if k is even since θ is a periodic parameter of period 2π. This is needed in order to have (1.1) as the effective twisted superpotential on the Coulomb branch [4]. The equation $t(v) = 1$ has an infinitely many solutions, and correspondingly, there are infinitely many supersymmetric vacua in this gauge system.

Each vacuum has a mass gap when, as assumed above, all the solutions to $t(v) = 1$ are non-degenerate. Things would be more interesting if $t(v)$ is fine tuned so that some of the solutions coincide, or equivalently, some of the solutions are degenerate. N solutions coincide, say at $v = 0$, when

$$t(v) = 1 + v^N + \cdots \quad \text{or} \quad P(v) = v^{N+1} + \cdots, \tag{1.4}$$

where the ellipses stand for possible terms of higher order in v. In such a case, we expect to have a non-trivial conformal field theory in the infra-red limit. In fact, for the case $k = 1$, it is argued in [5] that the vacuum at $v = 0$ is the same as the infra-red limit of the Landau-Ginzburg model with superpotential $W = X^{N+1}$, which is believed to be equivalent to the $\mathcal{N}{=}(2,2)$ superconformal minimal model of type A_{N-1}. For $k > 1$, we may have vacua where multiple M2 branes are at $v = 0$. We expect that all k of them can sit there as long as $N \geq k$. We would like to ask: What is the infra-red limit of such a theory?

We will argue that the theory under question is equivalent to the infra-red limit of the Landau-Ginzburg model of k variables $X_1, \ldots, X_k$, where the superpotential $W(X_1, \ldots, X_k)$ is $\mathrm{tr}\, \Sigma_T^{N+1}$ written in terms of the elementary symmetric functions of $\Sigma_1, \ldots, \Sigma_k$;

$$\sum_{a=1}^{k} \sigma_a^{N+1} = W(x_1, \ldots, x_k), \tag{1.5}$$

$$x_b = \sum_{a_1 < \cdots < a_b} \sigma_{a_1} \cdots \sigma_{a_b}, \qquad b = 1, \ldots, k \tag{1.6}$$

This model is believed to flow to the $\mathcal{N}{=}(2,2)$ superconformal Kazama-Suzuki model [6, 7] of the coset type

$$\frac{\mathrm{SU}(N)_1}{\mathrm{S}[\mathrm{U}(k) \times \mathrm{U}(N-k)]}. \tag{1.7}$$

Thus, we claim that the answer to the question is this Kazama-Suzuki model.

The behaviour (1.4) is realized simply by $t(v) = 1 + v^N$ or $P(v) = v^{N+1}$. In the former case, the 2d theory is the $\mathrm{U}(k)$ SQCD with N fundamental matter fields with fine tuned twisted masses [3, 5]. In the latter case, the 2d theory is the pure $\mathrm{U}(k)$ gauge theory with the tree level twisted superpotential

$$\mathcal{W} = \mathrm{tr}\, \Sigma^{N+1} + \pi i (k+1)\, \mathrm{tr}\, \Sigma. \tag{1.8}$$

For $1 \leq k \leq N$, we shall argue that the theory has, among infinitely many others, a set of ground states supported at $\Sigma = 0$, and this "$\Sigma = 0$ sector" flows to the superconformal field theory under question.

Thus, we have a purely field theoretical duality statement: for $1 \leq k \leq N$,

- the $\mathrm{U}(k)$ SQCD with N fundamentals having fine tuned twisted masses,
- the $\Sigma = 0$ sector of the pure $\mathrm{U}(k)$ gauge theory with superpotential (1.8), and
- the Landau-Ginzburg model with superpotential (1.5)

all flow to the infra-red fixed point given by the Kazama-Suzuki model (1.7).

The aim of this paper is to give evidence of the claims above. We compute the number of supersymmetric ground states and the chiral ring in the respective systems, and show that they agree. We also study the BPS spectrum of the brane system and compare it with the known field theoretical results. Superconformal points themselves are hard to analyze, and therefore we often make mass deformations. We also study the S^2 partition functions by using the recently-developed technique of exact computations [8–12] and show that they indeed agree.

The rest of the paper is organized as follows. In Section 2 and 3, we test our claim by studying supersymmetric ground states and the chiral ring. In Section 4, we study BPS solitions of the M2-brane system and compare the structure with the BPS spectrum of the Landau-Ginzburg theory [13]. If we take a certain limit, the BPS spectrum can be determined by using the technique of spectral networks [14]. In Section 5, we calculate the S^2 partition functions of the 2d theories on both sides of the claimed equivalence and show that they agree. The sections 4 and 5 can be read independently. In Appendix A, we briefly review the basic facts of Kazama-Suzuki models and their correspondence with Landau-Ginzburg theories. In Appendix B, we give a proof of some algebraic statement needed for the study of chiral ring. In Appendix C, we discuss the convergence of the integral appearing in the S^2 partition function.

2 Supersymmetric Vacua

As the first check, we look at the supersymmetric vacua of the respective systems or compute the Witten index [15], and see if the results are consistent with the claimed duality.

2.1 Brane System

Let us first look at the brane system. As in the introduction, we denote the set of solutions to $t(v) = 1$ by $\{v_i\}_{i \in I}$ where we initially assume that each solution is non-degenerate $t'(v_i) = 0$. Supersymmetry requires each M2-brane to have a fixed position in (t, v). The boundary at M5$'$ fixes t to be 1 and allowes v to be arbitrary, while the boundary at M5 requires the relation $t = t(v)$. Thus, each M2 must be at $t = 1$ and has $v = v_i$ for some $i \in I$. The s-rule requires different M2 branes to have different values of v. Therefore, a supersymmetric vacuum is specified by picking k distinct elements from this set:

$$V \subset \{v_i\}_{i \in I}, \quad |V| = k. \tag{2.1}$$

When $t(v)$ is a polynomial of order M, the equation $t(v) = 1$ has M roots. Generically, they are distinct and non-degenerate. Then, the number of supersymmetric vacua is 0 if $k > M$ and $\binom{M}{k}$ if $k \leq M$. We may also consider a special polynomial where some of the solutions coincide. In this situation we do not know how to identify the supersymmetric vacua. However, the Witten index [15], which does not change under continuous deformation, remains the same as in the non-degenerate case. When some number, say N, of the solutions are close to each other while others are far away, then, we may consider the "subsector" in which all k M2 branes are at one of these N solutions. In particular, when $N \geq k$ of them are at the same point, we expect to have a *single* infra-red theory whose Witten index is $\binom{N}{k}$. This discussion on subsectors and their Witten indices is applicable even when $t(v)$ is not a polynomial and the equation $t(v) = 1$ have infinitely many solutions.

2.2 Gauge Theory

Let us next consider the U(k) gauge theory with the tree level twisted superpotential (1.3) determined by a polynomial $P(\sigma)$. The classical scalar potential takes the form

$$U = \frac{1}{4g^2}\,\mathrm{tr}[\sigma, \sigma^\dagger]^2 + \frac{g^2}{2}\,\mathrm{tr}(\mathrm{Re}P'(\sigma))^2. \tag{2.2}$$

Vanishing of the first term requires σ to be diagonalizable,

$$\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_k). \tag{2.3}$$

When all the eigenvalues are well separated, the value of σ breaks the gauge group U(k) to its diagonal subgroup $T \cong \mathrm{U}(1)^k$. In this Coulomb branch, we may integrate out the off-diagonal components of the vector multiplet. This induces a correction to the twisted superpotential. As explained in [4] following [16], the correction is given by πi times the sum of positive roots,[1]

$$\Delta\mathcal{W} = \pi i \sum_{a<b} (\Sigma_a - \Sigma_b) \equiv \pi i(k+1) \sum_{a=1}^{k} \Sigma_a. \tag{2.4}$$

[1]Incidentally, this settles a problem concerning the relation observed in [17] between the tree level theta angle of the U(k) linear sigma model and the B-field of the corresponding non-linear sigma model on a complete intersection in the Grassmannian $G(k, N)$. The shift (2.4) was missed in [17] and the relation is mistakenly stated as "$B = \theta + (N + k + 1)\pi$". This must be corrected to $B = \theta + N\pi$. Now it is understandable as result from integrating out the "P-fields" [18].

In the second equality, we used the periodicity $\theta_a \equiv \theta_a + 2\pi$ of the theta angle for the group T. This cancels the tree level theta term in (1.3) and hence the effective twisted superpotential is

$$\mathcal{W}_{\text{eff}} = \mathcal{W}|_T + \Delta\mathcal{W} = \sum_{a=1}^{k} P(\Sigma_a). \tag{2.5}$$

We denote the effective gauge coupling constant by $e^2_{ab}(\sigma)$. We know that it approaches $g^2\delta_{ab}$ in the limit where all σ_a are infinitely separated. We assume that it is positive definite in the region of σ we are looking at, and defines inner products, $\|y\|^2_{e^{-2}} = (e^{-2})^{ab}y_a y_b$ and $\|x\|^2_{e^2} = e^2_{ab}x^a x^b$, on the Lie agbera of T and its dual. The effective potential is given by

$$U_{\text{eff}} = \frac{1}{2}\left\|\text{Re}\mathcal{W}'_{\text{eff}}(\sigma)\right\|^2_{e^2} + \frac{1}{2}\left\|v_{01}\right\|^2_{e^{-2}}. \tag{2.6}$$

The first term, where $\text{Re}(\mathcal{W}'_{\text{eff}}(\sigma))^a = \text{Re}P'(\sigma_a)$, is the remnant of the classical potential (2.2). The second term is the electro-static energy [19, 20]. In the Hamiltonian formulation, see e.g. [21], $(e^{-2})^{ab}v_{b01} + \text{Im}(\mathcal{W}'_{\text{eff}}(\sigma))^a$ are regarded as the conjugate momenta for the holonomy of T, each of which has period 1, and hence have eigenvalues in $2\pi\mathbb{Z}$. In other words,

$$v_{a01} = \sum_{b=1}^{k} e^2_{ab}(\sigma)\left(2\pi n^b - \text{Im}P'(\sigma_b)\right) \tag{2.7}$$

where $n^a \in \mathbb{Z}$. In the sector with definite n^a's, the effective potential is

$$U_{\text{eff}} = \sum_{a,b=1}^{k} \frac{e^2_{ab}(\sigma)}{2}(P'(\sigma_a) - 2\pi i n^a)\overline{(P'(\sigma_b) - 2\pi i n^b)}. \tag{2.8}$$

Supersymmetric ground states must be at the zeroes of this potential. That is, each (σ_a, n^a) must satisfy

$$P'(\sigma) = 2\pi i n, \qquad n \in \mathbb{Z}. \tag{2.9}$$

The above analysis is valid only when $\sigma_1, \ldots, \sigma_k$ are separated. We do not know how to analyze the region near the diagonals where some of σ_a's coincide. In many examples, however, it is found that no supersymmetric ground state is supported near the diagonals as long as the critical points of the effective twisted superpotential are all non-degenerate. See for example [17]. Here we assume that this applies to our system. Note also that solutions related by permutations of (σ_a, n^a)'s are related by the residual gauge transformations and must be identified. Thus, when $P''(\sigma) \neq 0$ at each solution to (2.9), a supersymmetric vacuum is specified by a choice of k unordered solutions $\{(\sigma_a, n^a)\}$ to (2.9) such that $\sigma_a \neq \sigma_b$ for $a \neq b$. We see that there are infinitely many supersymmetric vacua.

The equation (2.9) may be written simply as $\exp(P'(v)) = 1$. Then we see that the problem of finding supersymmetric vacua in this system is identical to that in the M2 brane system where the function $t(v)$ defining the M5 curve is given by (1.2).

Let us write

$$P_u(\sigma) = \frac{1}{N+1}\sigma^{N+1} + \sum_{j=1}^{N} \frac{u_j}{N+1-j}\sigma^{N+1-j}. \tag{2.10}$$

for which the equation (2.9) reads

$$\sigma^N + \sum_{j=1}^{N} u_j \sigma^{N-j} = 2\pi i n, \qquad n \in \mathbb{Z}. \tag{2.11}$$

For a small but generic $u = (u_1, \ldots, u_N)$, the equation with $n = 0$ has N distinct solutions close to $\sigma = 0$, while the equation with $n \neq 0$ has N separated solutions at $|v| \sim (2\pi n)^{1/N}$. Our main interest will be the sector with $n^1 = \cdots = n^k = 0$. The supersymmetric vacua must have σ_a values from the N solutions near 0. The number of such vacua is zero when $k > N$ and $\binom{N}{k}$ when $1 \leq k \leq N$. When we turn off u, the N solutions all go to $\sigma = 0$. If $1 \leq k \leq N$, we expect to have a single infra-red theory from the $n^1 = \cdots = n^k = 0$ sector. Its Witten index is $\binom{N}{k}$.

2.3 Landau-Ginzburg Model

Finally, we consider the Landau-Ginzburg model. Let $W_u(X) = W_u(X_1, \ldots, X_k)$ be the superpotential corresponding to $P_u(\sigma)$ of (2.10), that is, $\sum_{a=1}^{k} P_u(\Sigma_a)$ written in terms of the elementary symmetric functions of $\Sigma_1, \ldots, \Sigma_k$.

When we turn off u, the superpotential $W_0(X)$ is the one (1.5) given in the introduction and is a quasi-homogeneous polynomial. When $N \geq k$, it has an isolated critical point at $X = 0$ and the Landau-Ginzburg model is believed to flow to a non-trivial superconformal field theory of central charge $c = 3k(N-k)/(N+1)$. In fact the conformal field theory has been claimed to be equivalent to the Kazama-Sukuki supercoset of the type (1.7). See Appendix A. The space of supersymmetric ground states of the model is naturally identified with the representation $\wedge^k \mathbb{C}^N$ of $\mathrm{SU}(N)$ [22]. Its dimension $\binom{N}{k}$ matches the Witten index of the M2 and the gauge systems.

The model with $u_j \neq 0$ can be regarded as a perturbation of this superconformal field theory by the chiral primary fields $\phi_j(X)$ corresponding to $\sum_{a=1} \sigma_a^{N+1-j}$. These have R-charges $2(N+1-j)/(N+1)$ and conformal weigths $(N+1-j)/(N+1) < 1$ and hence the perturbation is relevant. In particular, the number of supersymmetric ground states remains the same, $\binom{N}{k}$. Moreover, for the particular deformation where all u_j but u_N vanish, the ground states are labelled by the weights of the representation $\wedge^k \mathbb{C}^N$ of $\mathrm{SU}(N)$ mentioned above [13]. This picture matches with the one for the M2 and the gauge systems if we regard the roots of $\sigma^N + u_N = 0$ as the weights of the representation $\mathbb{C}^N$. This observation will be important when we compare the spectra of BPS solitons.

For a generic choice of u, the correspondence of the ground states with those of the gauge system can be seen more explicitly. The map $\sigma \mapsto x(\sigma)$, defined by the elementary symmetric functions $x_1(\sigma), \ldots, x_k(\sigma)$ of $\sigma_1, \ldots, \sigma_k$, is regular away from the diagonals, since the Jacobi matrix has determinant

$$\det\left(\frac{\partial x_b}{\partial \sigma_a}\right)_{1\leq a,b\leq k} = \prod_{1\leq a<b\leq k}(\sigma_a - \sigma_b). \tag{2.12}$$

The singular values, i.e., the image of the diagonals, shall be called *the discriminant*. Let us write $f_u(\sigma_1, \ldots, \sigma_k) = \sum_{a=1}^k P_u(\sigma_a)$. Then, we have

$$f_u(\sigma) = W_u(x(\sigma)). \tag{2.13}$$

Taking the first derivatives, we obtain

$$\frac{\partial f_u}{\partial \sigma_a}(\sigma) = \sum_{b=1}^k \frac{\partial x_b}{\partial \sigma_a}(\sigma)\frac{\partial W_u}{\partial x_b}(x(\sigma)). \tag{2.14}$$

This means that "off the diagonals" critical points of $f_u(\sigma)$ modulo permutations of σ_a's are in one-to-one correspondence with "off the discriminant" critical points of $W_u(x)$. Taking one more σ derivative and computing the determinant, one sees that the Hessian of $f_u(\sigma)$ vanishes if $x(\sigma)$ is a critical point of $W_u(x)$ on the discriminant. Therefore, if all the critical points of $f_u(\sigma)$ are non-degenerate, then, all the critical points of $W_u(x)$, if there exist, are off the discriminant and also non-degenerate. (Note however that $f_u(\sigma)$ may have a non-degenerate critical point on the diagonal that does not correspond to a critical point of $W_u(x)$.) This establishes a one-to-one correspondence between the supersymmetric ground states of the $n^1 = \cdots = n^k = 0$ sector of the gauge system and those of the Landau-Ginzburg model, for a generic u so that $f_u(\sigma)$ is a Morse function. In particular, this is one way to see that the number of critical points of $W_u(X)$ is zero for $N < k$ and $\binom{N}{k}$ for $N \geq k$.

3 Chiral Rings

In this section, we shall study the chiral ring of the gauge system and compare the result with that of the Landau-Ginzburg model. We consider the $\mathrm{U}(k)$ gauge theory with tree level twisted superpotential

$$\mathcal{W} = f(\Sigma) + \pi i(k+1)\,\mathrm{tr}\,\Sigma \tag{3.1}$$

where $f(\Sigma)$ is an adjoint invariant polynomal of Σ. The effective twisted superpotential on the Coulomb branch is $\mathcal{W}_{\text{eff}} = f(\Sigma_T)$. We shall use the same notation

$f(\sigma) = f(\sigma_1, \ldots, \sigma_k)$ for that symmetric polynomial, and denote simply by $W(X)$ the corresponding superpotential, $f(\sigma) = W(x(\sigma))$. Just as in (2.14), we have

$$\frac{\partial f}{\partial \sigma_a}(\sigma) = \sum_{b=1}^{k} \frac{\partial x_b}{\partial \sigma_a}(\sigma) \frac{\partial W}{\partial x_b}(x(\sigma)). \tag{3.2}$$

We assume that $f(\sigma)$ is a Morse function. Then, $W(x)$ is also Morse, and supersymmetric ground states of the $n^1 = \cdots n^k = 0$ sector of the gauge system are in one-to-one correspondence with those of the Landau-Ginzburg model.

The chiral ring of the Landau-Ginzburg model is generated by the chiral variables $x_1, \ldots, x_k$ and the relations are generated by

$$0 \equiv \left\{ \mathbf{Q}_B, g^{a\bar{b}}(\overline{\psi}_{b-} - \overline{\psi}_{b+}) \right\} = \partial_{x_a} W(x), \qquad a = 1, \ldots, k. \tag{3.3}$$

Here $\mathbf{Q}_B$ is the relevant supercharge, $g^{a\bar{b}}$ is the Kähler metric that appears in the kinetic term, and $\overline{\psi}_{b\pm}$ are the fermionic components of the antichiral multiplet $\overline{X}_b$. Hence the chiral ring is isomorphic to the Jacobi ring,

$$\mathrm{Jac}(W) = \mathbb{C}[x_1, \ldots, x_k]/(\partial_{x_1} W(x), \ldots, \partial_{x_k} W(x)). \tag{3.4}$$

The twisted chiral ring of the gauge system is generated by gauge invariant polynomials of σ. In the low energy description on the Coulomb branch, they reduce to symmetric functions of $\sigma_1, \ldots, \sigma_k$. To find the relations, we note that

$$\left\{ \mathbf{Q}_A, (e^{-2})^{ab}(\overline{\lambda}_{b-} - \lambda_{b+}) \right\} = (e^{-2})^{ab}(D_b + i v_{b01}) \tag{3.5}$$

where $\mathbf{Q}_A$ is the relevant supercharge while $\overline{\lambda}_{b-}$, λ_{b+} and D_b are fermionic and auxiliary components of the twisted antichiral multiplet $\overline{\Sigma}_b$. The auxiliary fields D_b are constrained to be

$$(e^{-2})^{ab} D_b = -\mathrm{Re}\, \partial_{\sigma_a} f(\sigma). \tag{3.6}$$

We also have equations like (2.7):

$$(e^{-2})^{ab} v_{b01} = -\mathrm{Im}\, \partial_{\sigma_a} f(\sigma) + 2\pi n^a, \tag{3.7}$$

where n^a are integers labeling the momenta of the holonomy variables. Therefore the relations are $\partial_{\sigma_a} f(\sigma) \equiv 2\pi i n^a$.[2] Our main interest is the $n^1 = \cdots = n^k = 0$ sector. The relations are

$$\partial_{\sigma_a} f(\sigma) \equiv 0, \qquad a = 1, \ldots, k. \tag{3.8}$$

We shall also accept relations of the form

$$\sum_{a=1}^{k} \frac{F_a(\sigma)}{\Delta(\sigma)^{\ell}} \partial_{\sigma_a} f(\sigma) \equiv 0 \tag{3.9}$$

[2] In Eqns (3.5), (3.6), (3.7) some fermion bilinear terms are ignored to simplify the expression. However, these final relations are exact.

where $F_a(\sigma)$ are polynomials and $\Delta(\sigma)$ is the Vandermond determinant

$$\Delta(\sigma) := \prod_{1\le a<b\le k} (\sigma_a - \sigma_b). \tag{3.10}$$

We allow division by $\Delta(\sigma)$ because σ_a's are assumed to be separated from each other in the Coulomb branch. Let I_f be the ideal of the ring $\mathbb{C}[\sigma_1, \dots, \sigma_k]^{\mathfrak{S}_k}$ of symmetric polynomials consisting of polynomials that can be written in the form on the left hand side of (3.9). Then, the twisted chiral ring is

$$\mathbb{C}[\sigma_1, \dots, \sigma_k]^{\mathfrak{S}_k}/I_f. \tag{3.11}$$

When $f(\sigma)$ is generic so that $W(x)$ has only isolated and non-degenerate critical points (i.e. $W(x)$ is Morse), one can show that this is isomorphic to the Jacobi ring $\mathrm{Jac}(W)$.

The proof goes as follows. First, we have an isomorphism $\mathbb{C}[x_1, \dots, x_k] \cong \mathbb{C}[\sigma_1, \dots, \sigma_k]^{\mathfrak{S}_k}$ given by $\phi(x) \mapsto \phi(x(\sigma))$. It is enough to show that the ideal $I_W = (\partial_{x_1}W, \dots, \partial_{x_k}W)$ is mapped precisely to I_f under this isomorphism. That I_W is mapped into I_f is obvious in view of (3.2) and the definition of I_f. To show that the map $I_W \to I_f$ is surjective, let $\phi(x)$ be a polynomial so that $\phi(x(\sigma))$ belongs to I_f. Then, $\phi(x(\sigma))$ vanishes on "off the diagonals" critical points of $f(\sigma)$. Here we recall from the previous section that $\sigma \mapsto x(\sigma)$ gives one-to-one correspondence between "off the diagonals" critical points of $f(\sigma)$ modulo permutations and critical points of $W(x)$. Therefore, $\phi(x)$ vanishes on the critical points of $W(x)$. Since $W(x)$ is a Morse function, this means that $\phi(x)$ belongs to I_W. See Appendix B for the proof of the last statement.

4 BPS Solitons

In this section we analyze the spectrum of the BPS states from M2-branes, building on [3, 23–26], and compare the results with the spectrum of BPS solitons in the Landau-Ginzburg model [13, 27].

In what follows, we are interested in M2-branes whose (t, v) values are confined into a small neighborhood of $v = 0$ and $t = 1$. Therefore, we write $t = e^z$ and regard z as a coordinate on a neighborhood of the origin of of a complex plane $\mathbb{C}$. M5$'$ is at $z = 0$ and we consider the M5-brane wrapped on the curve

$$z = v^N + u_1 v^{N-1} + \cdots + u_N. \tag{4.1}$$

Recall (2.1) that a supersymmetric ground state is specified for a choice of k distinct elements from the set $\{v_i\}_{i=1}^N$ of solutions to $v^N + u_1 v^{N-1} + \cdots + u_N = 0$. We are interested in solitonic M2-brane configurations that interpolate two different ground states.

4.1 A single M2-brane

Let us recall the basics of BPS solitons arising from a single M2-brane stretched between two M5-branes. This setup was originally studied in [3] and later in [26]. The system may be regarded as an $\mathcal{N}$=(2,2) supersymmetric field theory on $\mathbb{R}^2 = \{(x^0, x^1)\}$ with a chiral multiplet taking values in the space of paths $\phi : x^6 \in [0, L] \mapsto (z(x^6), v(x^6)) \in \mathbb{C}^2$ from the M5′ at $z = 0$ to the M5 at (4.1). It has the superpotential [28]

$$\mathcal{W}[\phi] = \int_{C_{\phi,\phi_*}} \Omega, \qquad \Omega := \mathrm{d}z \wedge \mathrm{d}v, \tag{4.2}$$

where C_{ϕ,ϕ_*} is a configuration that interpolates a reference path ϕ_* and ϕ. Note that

$$\frac{\delta\mathcal{W}}{\delta z(x^6)} = \partial_6 v(x^6), \qquad \frac{\delta\mathcal{W}}{\delta v(x^6)} = -\partial_6 z(x^6). \tag{4.3}$$

In particular, the action includes the usual kinetic term of a theory on three dimensions (x^0, x^1, x^6). A soliton is a configuration that approaches two vacua, say $\phi_j \equiv (0, v_j)$ and $\phi_i \equiv (0, v_i)$, as $x^1 \to -\infty$ and $x^1 \to +\infty$ respectively. The central charge of such a solitonic sector is

$$Z_{ij} = \mathcal{W}[\phi_i] - \mathcal{W}[\phi_j] = \int_{C_{\phi_i,\phi_j}} \Omega. \tag{4.4}$$

A soliton preserves a half of the supersymmetry if the configuration satisfies the BPS equation, $\partial_1\phi = \zeta_{ij}\overline{\delta\mathcal{W}/\delta\phi}$ with $\zeta_{ij} := Z_{ij}/|Z_{ij}|$, i.e.,

$$\partial_1 z = \zeta_{ij}\partial_6\overline{v}, \qquad \partial_1 v = -\zeta_{ij}\partial_6\overline{z}. \tag{4.5}$$

It follows that

$$\phi^*\omega = 0, \qquad \overline{\zeta}_{ij}\phi^*\Omega = \mathrm{d}x^1 \wedge \mathrm{d}x^6 \cdot (\text{real positive}), \tag{4.6}$$

where $\omega := \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z} + \frac{i}{2}\mathrm{d}v \wedge \mathrm{d}\overline{v}$ is the Kähler form. This is equivalent [24] to the condition that the image of $\phi : \mathbb{R} \times [0, L] \to \mathbb{C}^2$ is a special Lagrangian submanifold.

One may also look at the usual supersymmetry condition [23, 26, 29]. Let η be the eleven-dimensional spinor obeying $\eta = \Gamma_{012\ldots9,10}\eta$. Presence of the M5-branes imposes the condition $\eta = \Gamma_{014589}\eta = \Gamma_{012345}\eta = \Gamma_{01789,10}\eta$ from which we also have $\eta = \Gamma_{016}\eta$. Then, the BPS equation (4.5) is equivalent to the existence of a spinor η obeying

$$\eta = \frac{1}{2}\epsilon^{\alpha\beta}\Gamma_{0IJ}\partial_\alpha x^I \partial_\beta x^J \eta \tag{4.7}$$

(the summation over $I, J = 1, 4, 5, 6, 7, 10$ and $\alpha, \beta = 1, 6$ is assumed), in the limit $|\partial_{1,6}x^{4,5,7,10}| \ll 1$ where the eleven dimensional Planck length is set equal to one. The preserved supersymmetry is $(\zeta_{ij}\Gamma_{vz} + \overline{\zeta}_{ij}\Gamma_{\overline{v}\overline{z}})\eta = \Gamma_{16}\eta$.

If we change the complex structure of $\mathbb{C}^2$ so that $\omega - i\mathrm{Im}(\overline{\zeta}_{ij}\Omega)$ is a holomorphic two form and $\mathrm{Re}(\overline{\zeta}_{ij}\Omega)$ is a Kähler form, the BPS equation (4.5) has a different interpretation: It is a Cauchy-Riemann equation with respect to the holomorphic coordinate $x^1 + ix^6$ of the domain $\mathbb{R} \times [0, L]$. That is, $\phi : \mathbb{R} \times [0, L] \to \mathbb{C}^2$ can be regarded as a holomorphic map. With respect to the new Kähler form, the two M5 curves, $z = 0$ and (4.1), are Lagrangian submanifolds of $\mathbb{C}^2$ which intersects at the N points $\{(0, v_i)\}_{i=1}^N$. Therefore, a BPS solition from $(0, v_j)$ to $(0, v_i)$ can be identified as a term of the Floer differential of the pair of Lagrangian submanifolds, in the intersection Floer theory [30].

4.2 k M2-branes

We now consider the case of general k. Let us take two ground states specified by subsets V and V' of $\{v_i\}_{i=1}^N$ of order k. A soliton that interpolates V and V' is the superposition of k single M2-brane solitions, each of which approach $v_{i_a} \in V$ and $v_{i'_a} \in V'$ as $x^1 \to -\infty$ and $x^1 \to +\infty$ respectively. The central charge of such a solitonic sector is the sum $\sum_{a=1}^k Z_{i'_a, i_a}$ while the mass is bounded below by $\sum_{a=1}^k |Z_{i_a, i'_a}|$. When u_j are generic, Z_{ij} have different phases for different pairs (i, j). Therefore, it saturates the BPS bound only when just one of the k M2-branes is a non-trivial soliton while the remaining $k - 1$ stay fixed at the vacua. This is possible only when $|V \cap V'| = k - 1$.

In the picture where $\{v_i\}_{i=1}^N$ is regarded as the set of weights of the fundamental representation $\mathbb{C}^N$ of $\mathrm{SU}(N)$, a BPS state for the k M2-brane system exists only if V and V', which are regarded as weights of the representation $\wedge^k\mathbb{C}^N$, are connected by a root of $\mathrm{SU}(N)$. This matches with the structure of the BPS spectrum of the Landau-Ginzburg model: In [13, 27], it was proposed that there is exactly one BPS solition for each pair of vacua labelled by weights of $\wedge^k\mathbb{C}^N$ that differ by a root of $\mathrm{SU}(N)$. Therefore, we would like to see that there is exactly one BPS soliton for any pair of v_i and v_j in the single M2-brane system.

Showing this seems to be a difficult problem to the authors. Instead of trying to find BPS configurations directly, we shall take a certain limit [25] that reduces the problem of finding BPS membranes to the problem of finding BPS geodesics [31], and then use the technique of spectral networks.

4.3 BPS states via spectral networks

So far, we have been using the metric $\mathrm{d}s^2 = |\mathrm{d}z|^2 + |\mathrm{d}v|^2$ in the $x^{4,5,7,10}$ directions. We now change it to

$$\mathrm{d}s^2 = |\mathrm{d}z|^2 + \beta^2|\mathrm{d}v|^2 \tag{4.8}$$

and take a small β limit. We also have

$$\omega = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z} + \frac{i}{2}\beta^2\mathrm{d}v \wedge \mathrm{d}\overline{v}, \qquad \Omega = \beta\mathrm{d}z \wedge \mathrm{d}v. \tag{4.9}$$

The argument of [25] shows that, in the limit $\beta \to 0$, the projection of a BPS configuration $C_{ij} = C_{\phi_i,\phi_j}$ onto the z-plane is a real one-dimensional graph γ_{ij}, and the tangent directions Δz and Δv obey the constraint $\Delta z \cdot \Delta v = \zeta_{ij}\cdot$ (real number). Over a generic point z on the graph γ_{ij}, C_{ij} is a line segment from one solution v_l to another v_k of (4.1). Of course, $(k, l) = (i, j)$ near $z = 0$, but that may not be the case if z is far from $z = 0$. See Figure 2.

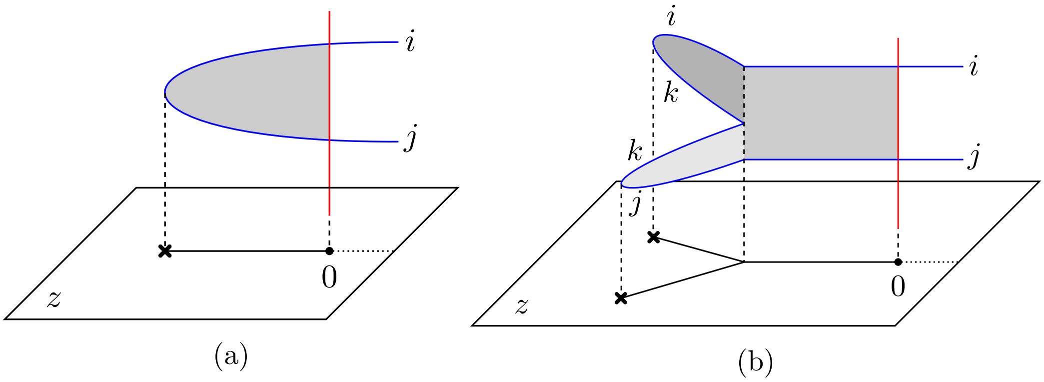


Figure 2: M2-brane solitons in the $\beta \to 0$ limit. Blue curves are parts of M5, and red lines are parts of M5′. The (z, v) images of the M2-brane solitons are shaded.

In a neighborhood of such a point, the graph γ_{ij} is a curve determined by the differential equation

$$\lambda_{kl}(z)\frac{\partial z}{\partial \tau} = \exp(i\vartheta_{ij}) = \frac{Z_{ij}}{|Z_{ij}|}, \tag{4.10}$$

where $\lambda_{kl}\mathrm{d}z = (v_k(z) - v_l(z))\mathrm{d}z$ is the difference of $\lambda = v\mathrm{d}z$ at the k-th sheet and at the l-th sheet of M5-branes, and τ is a real parameter along the curve γ_{ij}. Such a γ_{ij} is called a finite open web of BPS strings [14].

We would like to find a solution to (4.10) which starts from a branch point of the covering $v(z) \mapsto z$ and call it $\mathcal{S}_{kl}$. For a generic value of $\vartheta = \vartheta_{ij}$, it does not pass the endpoint of the ground-state M2-branes at $x = 0$ but goes to infinity, meaning that it does not correspond to any of the BPS states. These paths are called $\mathcal{S}$-walls. When two $\mathcal{S}$-walls $\mathcal{S}_{ik}$ and $\mathcal{S}_{kj}$ cross, another $\mathcal{S}$-wall, $\mathcal{S}_{ij}$, can emerge, when there is a supersymmetric junction of three M2-branes that satisfy $\lambda_{ik} + \lambda_{kj} = \lambda_{ij}$ [14, 32], like in Figure 2b. The collection of $\mathcal{S}$-walls is called a spectral network [14]. When there is an $\mathcal{S}$-wall $\mathcal{S}_{ij}$ that passes $z = 0$, then this gives us a BPS object with a finite central charge.

4.3.1 Deformation by u_N

Let us consider a particular deformation where the curve is

$$z = v^N + u_N. \tag{4.11}$$

The z-coordinate is zero when

$$v_j = (-u_N)^{1/N}\omega^j, \quad \text{where} \quad \omega = e^{2\pi i/N}. \tag{4.12}$$

Hence, the vacua are depicted by the vertex of a regular polygon on the v-plane.

The curve has a branch point of ramification index N at $z = u_N$, and the differential equation that governs the behavior of each $\mathcal{S}_{ij}$ on the z-plane is

$$\alpha_{ij}(z-u_N)^{1/N}\frac{\partial z}{\partial \tau} = \exp(i\vartheta), \tag{4.13}$$

where $\alpha_{ij} = \omega^i - \omega^j$. The solution is

$$z_{ij}(\tau) = u_N + c\left(\frac{N+1}{N}\frac{\tau}{\alpha_{ij}}\right)^{N/N+1}\exp\left(\frac{N}{N+1}i\vartheta\right) \tag{4.14}$$

where c is an $(N+1)$-st root of unity. This is a straight line starting at the branch point $z = u_N$. When $\alpha_{ij} = \alpha_{i'j'}$ two S-walls can be on top of each other. As an example, Figure 3 shows the spectral network when $N = 4$ for $\vartheta = 0$. As can be seen there, $\mathcal{S}_{12}$ and $\mathcal{S}_{34}$ are coincident.

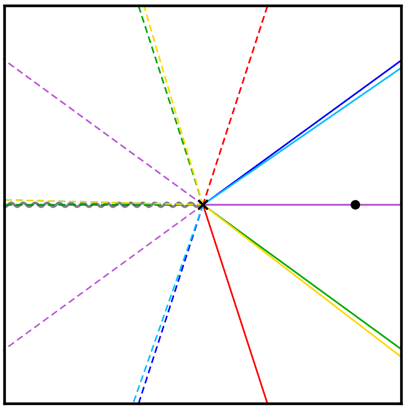
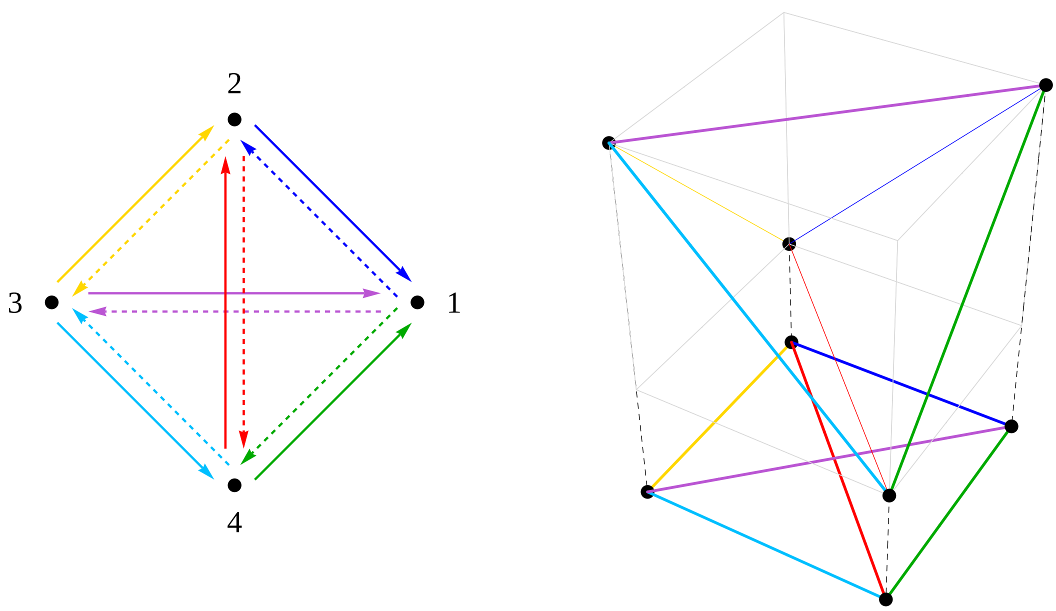


Figure 3: A spectral network around a branch point of ramification index $N = 4$.

When we change ϑ from 0 to 2π, the whole spectral network rotates by $2\pi N/(N+1)$, and the endpoint of the M2-brane meets $N(N-1)$ $\mathcal{S}$-walls in the process, implying there are in total $N(N-1)$ BPS states in the BPS spectrum of this theory. Therefore, for each distinct i and j, there is one BPS state in the sector with the right boundary set to the vacuum i and the left boundary set to the vacuum j. It is easy to identify the value ϑ when an $\mathcal{S}_{ij}$ wall hits $x = 0$. There is one value of ϑ for each $\mathcal{S}_{ij}$.

On the v-plane, we can introduce a soliton of the $k = 1$ theory by a line connecting v_i and v_j. Let us illustrate the case $N = 4$. Figure 4a represents the four ground states and twelve solitons on the v-plane. We clearly see that $Z[\gamma_{12}]$ and $Z[\gamma_{34}]$ has the same phase, as was also reflected in the spectral network shown in

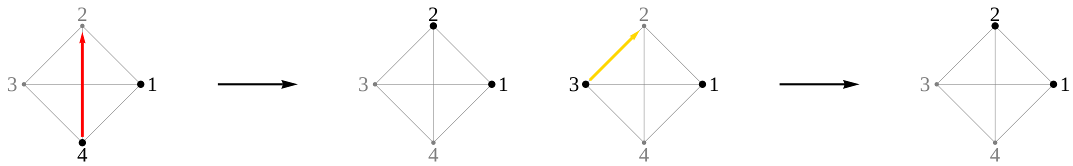


(a) on the v-plane

(b) in the weight lattice of SU(4)

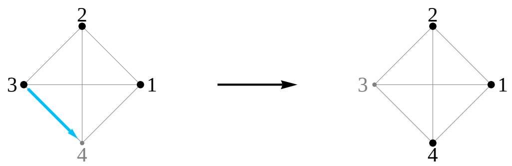
Figure 4: Vacua and solitons, $k = 1$.

Figure 3. Note that Figure 4a can be understood as obtained from the projection of the weights of the fundamental representation of SU(4) and the roots connecting the weights, representing the ground states and the solitons respectively, as shown in Figure 4b. This structure of BPS solitons is the same as that of the corresponding Landau-Ginzburg model with a single chiral field, which has as its IR fixed point the $\mathcal{N}$=2 A_3 minimal model [27].



(a) [14] $\to$ [12] soliton for $k = 2$

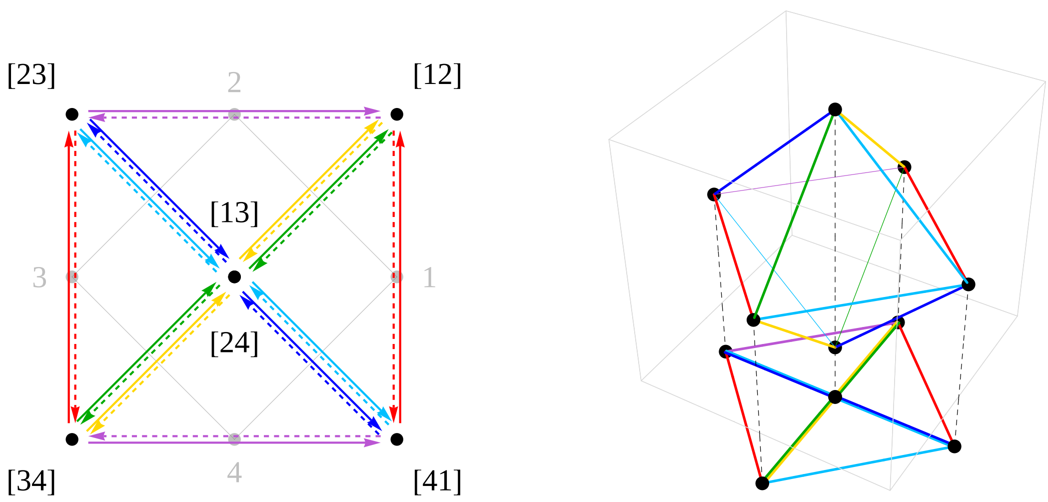
(b) [13] $\to$ [12] soliton for $k = 2$



(c) [123] $\to$ [412] soliton for $k = 3$

Figure 5: Examples of $k > 1$ solitons.

So far we discussed the case when there is just one M2-brane, $k = 1$. For general k, we need to choose k vertices out of N, and a soliton is obtained by moving one of the k vertices. In Figure 5, some representative examples of the solitons with $k = 2$ and $k = 3$ are shown. For $k = 2$, we see from Figures 5a and 5b that a $k = 1$ solitonic configuration can connect two $k = 2$ ground states. From this consideration we can represent $k = 2$ ground states and solitons as shown in Figure 6a. Again, we can understand this as obtained from the projection of the weights of the 2nd antisymmetric power of the fundamental representation of SU(4) and the roots connecting the weights, as shown in Figure 4b. The same structure of BPS solitons of the corresponding Landau-Ginzburg model is observed in [13], which is expected to flow in the IR to the Kazama-Suzuki model based on $\mathrm{SU}(4)_1/\mathrm{S}[\mathrm{U}(2) \times \mathrm{U}(2)]$.



(a) on the v-plane

(b) in the weight lattice of SU(4)

Figure 6: Vacua and solitons, $k = 2$.

For $k = 3$, because choosing k ground states among N indistinguishable ones is the same as choosing $N - k$ ground state, the ground states and the solitons are represented by the same diagram as Figure 4a, thus we see the $k \leftrightarrow N - k$ duality.

4.3.2 General deformations

Now let us consider how the spectral networks look when the deformation parameters u_j are general.

BPS spectrum with $z = v^3$

Now we consider the case where we have three M5-branes ramified over the z-plane:

$$z = v^3 + u_2 v + u_3. \tag{4.15}$$

For general u_2 and u_3, we have two branch points of ramification index 2 on the z-plane as shown in Figure 7. In the figure, we chose $u_{2,3}$ so that a (12)-branch cut, a blue wavy line, comes out from the upper branch point, and (13)-branch cut, a green wavy line, from the lower branch point. From the (12)-branch point we have three $\mathcal{S}$-walls: two $\mathcal{S}_{21}$ with solid blue line and one $\mathcal{S}_{12}$ with a dashed blue line. Similarly, from the (13)-branch point, we have two $\mathcal{S}_{13}$ with solid green line and one $\mathcal{S}_{31}$ with dashed green line. We can see that one $\mathcal{S}_{21}$ and one $\mathcal{S}_{13}$ meet at a point, from which another $\mathcal{S}$-wall, $\mathcal{S}_{23}$, emerges.

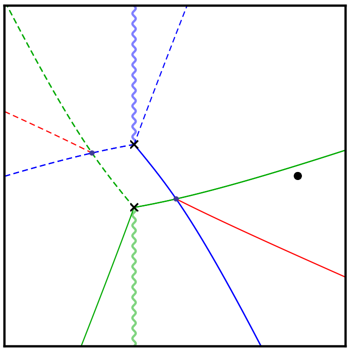
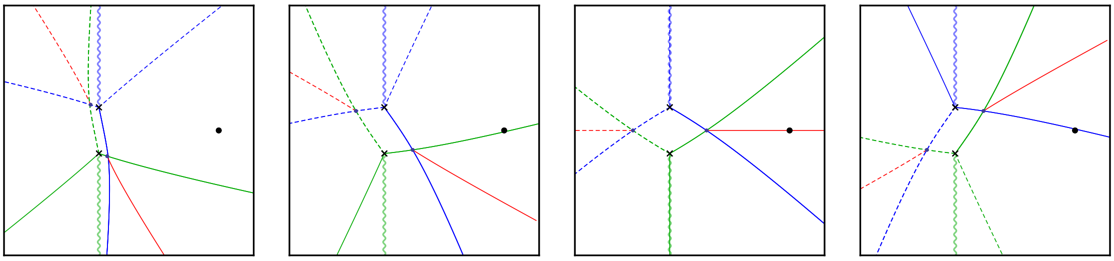


Figure 7: A spectral network with general u_2 and u_3.



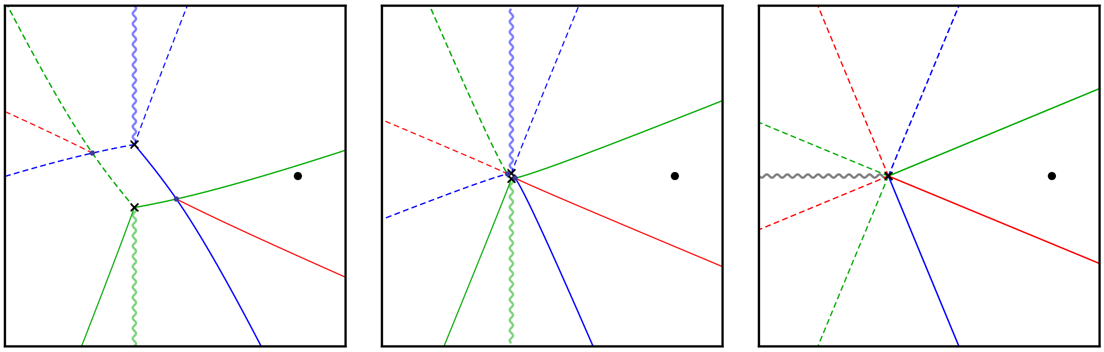
(a) $\vartheta \approx 0$ (b) $\vartheta \approx \vartheta_{13}$ (c) $\vartheta \approx \vartheta_{23}$ (d) $\vartheta \approx \vartheta_{21}$

Figure 8: Rotation of a spectral network with general u_2 and u_3.

As we now have the full spectral network, let us rotate it by changing ϑ from 0 to 2π. Figure 8 shows spectral networks at various values of ϑ, $0 < \vartheta_{13} < \vartheta_{23} < \vartheta_{21} < \pi$. We see that there are γ_{13}, γ_{23}, and γ_{21} at ϑ_{13}, ϑ_{23}, and ϑ_{21}, respectively, between the branch point and the M2-brane endpoint. Therefore there are corresponding three BPS states for $0 < \vartheta < \pi$. There are another three BPS states for $\pi < \vartheta < 2\pi$, each of which has the central charge $Z[\gamma_{ji}] = -Z[\gamma_{ij}]$.

Now let us take the limit $u_2 \to 0$ so that the two branch points collide, see Figure 9. Figure 9c shows the spectral network when $u_2 = 0$. There is only a single (123)-branch point and a single (123)-branch cut.[3] The whole spectral network rotates

[3]The notation is that around the $(n_1 n_2 \cdots n_k)$ branch cut the sheets are exchanged in the order $n_1 \to n_2 \to \cdots \to n_k \to n_1$.



(a) $u_2 \approx 1$ (b) $u_2 \ll 1$ (c) $u_2 = 0$

Figure 9: Evolution of the spectral network under the limit of $u_2 \to 0$.

by $3\pi/4$ when we change ϑ from 0 to π continuously, and in the process we find three BPS strings connecting the branch point and the endpoint of the M2-brane, corresponding to three BPS states in $0 < \arg(Z) < \pi$.

BPS spectrum with $z = v^4$

Let us now consider the case $N = 4$, for more illustration.

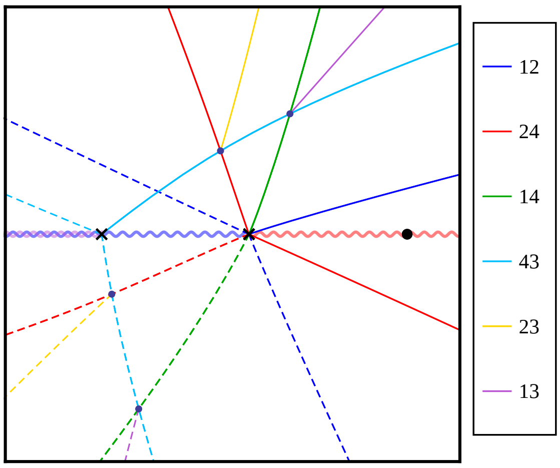
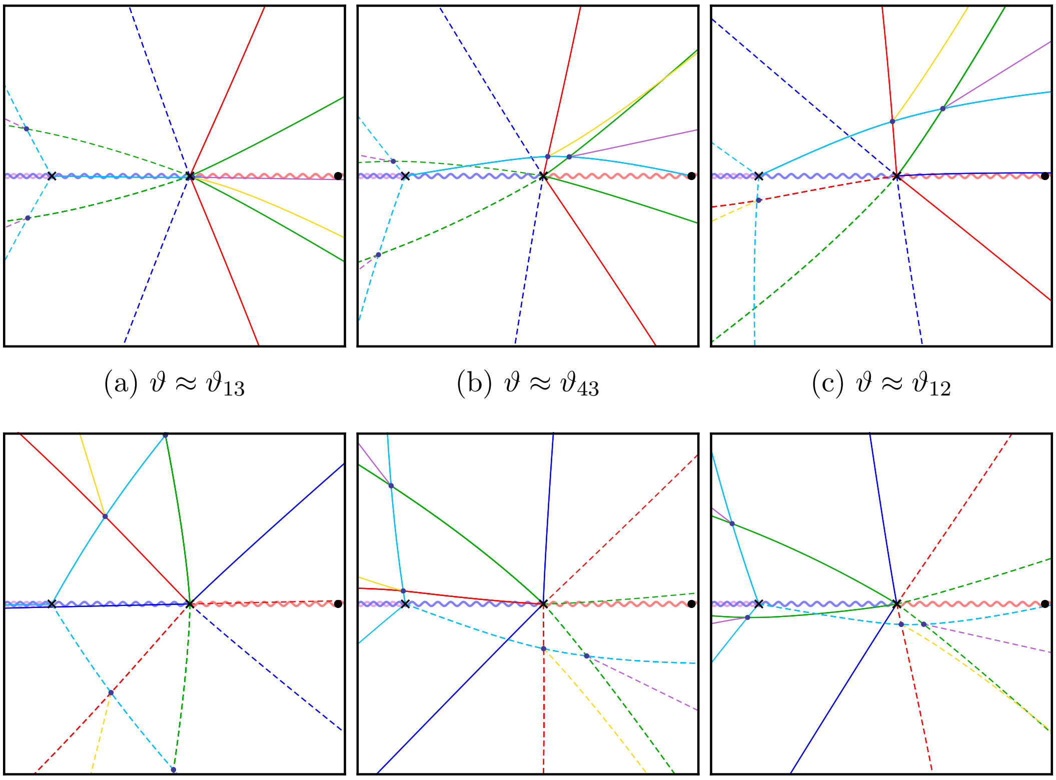


Figure 10: A spectral network with $u_{2,3,4} \neq 0$.

Figure 10 shows the spectral network with $u_{2,3,4}$ chosen so that there is a (124)-branch point and a (34)-brach point. See the legend for the nature of walls represented by the colors and the styles.

Figure 11 shows the spectral network at various values of ϑ. At ϑ_{ij} there is γ_{ij} between one of the branch points and the endpoint of the M2-brane, and Figure 11 is arranged such that

$$0 < \vartheta_{13} < \vartheta_{43} < \vartheta_{12} < \vartheta_{42} < \vartheta_{41} < \vartheta_{34} < \pi. \tag{4.16}$$

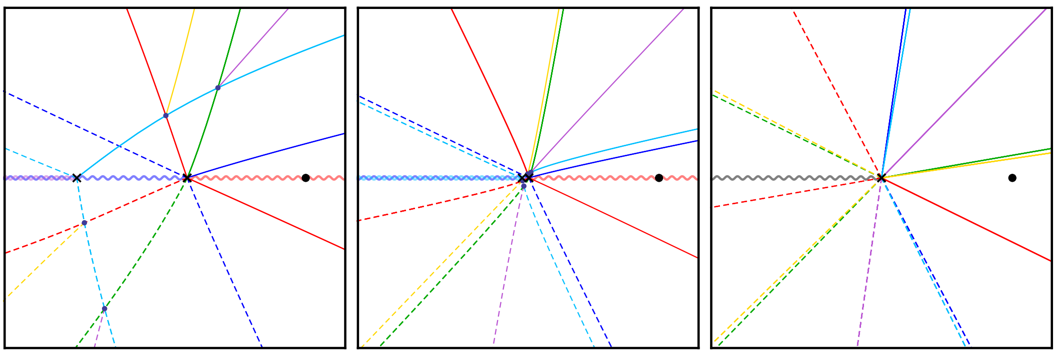


(a) $\vartheta \approx \vartheta_{13}$ (b) $\vartheta \approx \vartheta_{43}$ (c) $\vartheta \approx \vartheta_{12}$

(d) $\vartheta \approx \vartheta_{42}$ (e) $\vartheta \approx \vartheta_{41}$ (f) $\vartheta \approx \vartheta_{34}$

Figure 11: Rotation of the spectral network with $u_{2,3,4} \neq 0$.

That is, we can imagine the whole spectral network rotating anti-clockwise as we increase ϑ from 0 to π and in the course of the rotation we encounter six finite BPS strings. Therefore, we can expect this theory to have twelve BPS states in total. There are four vacua, so there is one BPS state for each boundary condition at the left and the right spatial infinity.



(a) $|u_2| \sim 1$ (b) $|u_2| \ll 1$ (c) $u_2 = 0$

Figure 12: Evolution of the spectral network under the limit of $u_{2,3} \to 0$.

Let us consider what happens when we have just one branch point of ramification index 4. This limit corresponds to $u_2, u_3 \to 0$, and the evolution of the spectral network under the limit is depicted in Figure 12. Thus we see that the spectrum of the BPS states at general $u_{2,3,4} \neq 0$ is smoothly connected to the more symmetric situation analyzed in Sec. 4.3.1 with $u_{2,3} = 0$ and $u_4 \neq 0$.

5 S^2 partition functions

As a final check of our proposal, we show in this section that the partition function on S^2 of the 2d $\mathcal{N}$=(2,2) U(k) gauge theory with twisted superpotential $\mathcal{W} = \operatorname{tr} P(\Sigma) + \pi i(k+1)\operatorname{tr}(\Sigma)$ in the infrared limit agrees with that of the Landau-Ginzburg model with chiral fields $X_1, \ldots, X_k$ with appropriately chosen superpotential $W = W(X_1, \ldots, X_k)$. We employ the localization methods recently developed in [8–10]. The derivation can be easily generalized to arbitary gauge group, and the quasihomogeneity of P and W is not required, either. The integrals below are only conditionally convergent. In this section we perform the comparison of the partition functions rather naively. The convergence issues will be explained in Appendix C. It will be then clear that the manipulations can be readily justified.

The partition function of the Landau-Ginzburg model of k variables $X_1, \ldots, X_k$ with the superpotential $W(X_1, \ldots, X_k)$ is given by [10]

$$Z_{\text{LG}} = (r\Lambda)^k \int_{\mathbb{C}^k} \prod_a dX_a d\overline{X}_a e^{-ir[W(X)+\overline{W}(\overline{X})]}. \tag{5.1}$$

where r is the radius of the sphere. The factor in front, $(r\Lambda)^k$, with Λ being a renormalization scale, was not explicitly in [8–10] but its presence is mentioned in a footnote of [9] and the computation was done by the authors of these papers [33, 34]. The same applies to $(r\Lambda)^{-k^2}$ in (5.2) below. See [12] for a detailed explanation in a related context. When W is quasi-homogeneous, a rescaling of fields can absorb the r in the integrand and yields the expected behaviour $Z_{\text{LG}} \sim r^{\hat{c}}$ with $\hat{c}$ being the expected central charge of the infra-red fixed point of the model [35, 36].

The partition function of the $\mathcal{N}$=(2,2) supersymmetric gauge theory was first computed in [8, 9] up to a sign factor which was later corrected in [11, 12]. The one for the theory with gauge group U(k) and with the twisted superpotential $\mathcal{W}(\Sigma)$ is given by

$$Z_{\text{gauge}} = (r\Lambda)^{-k^2} \sum_{m\in\mathbb{Z}^k} \int_{\mathbb{R}^k} \prod_a d(r\tau_a) \prod_{a<b} \left(r^2(\tau_a - \tau_b)^2 + \frac{(m_a - m_b)^2}{4} \right) \\ \times (-1)^{(k+1)\sum_a m_a} e^{-ir[\mathcal{W}(\Sigma)+\overline{\mathcal{W}}(\overline{\Sigma})]} \tag{5.2}$$

where

$$\Sigma = \operatorname{diag}(\tau_1, \ldots, \tau_k) + \frac{i}{2r} \operatorname{diag}(m_1, \ldots, m_k) \tag{5.3}$$

in the exponent.[4] For the twisted superpotential $\mathcal{W}(\Sigma) = \mathrm{tr}P(\Sigma) + \pi i(k+1)\mathrm{tr}(\Sigma)$, the formula (5.2) reads

$$Z_{\text{gauge}} = \Lambda^{-k^2} \sum_{m\in\mathbb{Z}^k} \int_{\mathbb{R}^k} \prod_a d\tau_a \prod_{a<b} \left((\tau_a - \tau_b)^2 + \left(\frac{m_a - m_b}{2r}\right)^2 \right) e^{-ir[\mathrm{tr}P(\Sigma)+\mathrm{tr}\overline{P}(\overline{\Sigma})]} \tag{5.4}$$

Now, look at the infra-red regime $r\Lambda \gg 1$. The sum $\sum_{m\in\mathbb{Z}^k}$ in (5.4) turns into an integral $(2r)^k \int_{\mathbb{R}^k} \prod_a dv_a$ for $v_a = \frac{m_a}{2r}$, and we have

$$Z_{\text{gauge}} \overset{r\Lambda\gg 1}{\longrightarrow} \Lambda^{-k^2} r^k \int_{\mathbb{C}^k} \prod_a d\sigma_a d\overline{\sigma}_a \prod_{a<b} |\sigma_a - \sigma_b|^2 e^{-ir[\mathrm{tr}\,P(\Sigma)+\mathrm{tr}\,\overline{P}(\overline{\Sigma})]} \tag{5.5}$$

where $\sigma_a = \tau_a + iv_a$ and

$$\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_k). \tag{5.6}$$

Let us introduce variables X_a as the elementary symmetric polynomials of σ_a; equivalently, let us take

$$\det(z - \Sigma) = \sum_a X_a z^{k-a}. \tag{5.7}$$

where z is a dummy variable. Then the Jacobian between the variables σ_a and the variables X_a are given as in (2.12),

$$\det\left(\frac{\partial X_b}{\partial \sigma_a}\right)_{1\le a,b\le k} = \prod_{1\le a<b\le k} (\sigma_a - \sigma_b). \tag{5.8}$$

Therefore, we see that the gauge partition function in the infrared, (5.5), agrees with the Landau-Ginzburg partition function (5.1), under the identification

$$P(\Sigma) = W(X_1, \dots, X_k). \tag{5.9}$$

We now have the equality of S^2 partition functions of the U(k) theory with the twisted superpotential in the infrared limit and those of the Landau-Ginzburg theory. Two-point functions of BPS operators can be dealt with in the completely same way, by just inserting the operators in the integral. It is well-known that the resulting integral expressions suffer from subtleties: apparently spurious operators do not decouple and the choice of representatives of the (anti)chiral ring elements matters [37]. The agreement holds provided that the operators in the gauge system are identified with those in the Landau-Ginzburg model precisely via the isomorphism $\mathbb{C}[\sigma_1, \dots, \sigma_k]^{\mathfrak{S}_k} \cong \mathbb{C}[X_1, \dots, X_k]$.

[4]The sign factor $(-1)^{(k+1)\sum_a m_a}$ was not in [8, 9]. Its presence only changes the weight of the sum over the topological type of the U(k) gauge bundle, only when k is even. Therefore such a factor is rather sutble. The presence is demanded for the factorization of the sphere partition function into two hemispheres [11, 12]. As we will see, its presence is also needed for the match with the partition function of the proposed Landau-Ginzburg model.

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A On Kazama-Suzuki models and their Landau-Ginzburg descriptions

The Kazama-Suzuki model [6] is a coset model G_k/H where G is a compact simple simply connected Lie group, H is its closed subgroup of the same rank as G such that the space G/H of left cosets is Kähler; k is a positive integer. It can be realized as a gauge theory [38]: the gauge group is H/Z_G (Z_G is the center of G) and the matter theory is the direct product of the G_k Wess-Zumino-Witten model and the $\mathfrak{g}/\mathfrak{h}$-valued free fermion, where H acts on G and $\mathfrak{g}/\mathfrak{h}$ by the conjugation. The models relevant for us are a subclass of

$$\frac{\mathrm{SU}(m+n)_k}{\mathrm{S}[\mathrm{U}(m)\times\mathrm{U}(n)]} \tag{A.1}$$

with the central charge

$$\hat{c} = \frac{c}{3} = \frac{kmn}{k+m+n}. \tag{A.2}$$

This model is invariant under permulations of k, m, n [6]. The model with $m = n = 1$, i.e. $\mathrm{SU}(2)_k/\mathrm{U}(1)$, is equivalent to the $\mathcal{N} = 2$ A_k minimal model [39]. The model with $m = 1$, $n = N - k$, i.e.

$$\frac{\mathrm{SU}(N)_1}{\mathrm{S}[\mathrm{U}(k)\times\mathrm{U}(N-k)]}, \tag{A.3}$$

is believed [22, 40] to be equivalent to the IR fixed point of a Landau-Ginzburg model with a superpotential $W(x_1, \ldots, x_k)$ which is chosen so that

$$W(x_1, \ldots, x_n) = \sum_{b=1}^{k} {\sigma_b}^N, \tag{A.4}$$

where σ_b are auxiliary variables such that x_b are their elementary symmetric polynomials:

$$x_b = \sum_{1 \le l_1 < l_2 < \cdots < l_b \le k} \sigma_{l_1} \sigma_{l_2} \cdots \sigma_{l_b}. \tag{A.5}$$

One piece of evidence of the equivalence comes from computing the central charge and the spectrum of the operators on each side and matching them. In addition, when $k > N-k$, we can re-express everything in terms of $N-k$ chiral fields, which implies $k \leftrightarrow N-k$ duality [40]. Another nontrivial evidence comes from the calculation of elliptic genera in the two descriptions, which yields agreement [41, 42].

B Some algebra

We show that, when $W(x)$ is a Morse polynomial of k variables, $x = (x_1, \ldots, x_k)$, a polynomial $\phi(x)$ that vanishes at all the critical points of $W(x)$ belongs to the ideal $I_W = (\partial_{x_1} W(x), \ldots, \partial_{x_k} W(x))$. [5] We have the following exact sequence of sheaves of $\mathcal{O}$ modules on $\mathbb{C}^k$, where $\mathcal{O}$ is the sheaf of algebraic functions of $x_1, \ldots, x_k$:

$$0 \to \mathcal{I}_W \to \mathcal{O} \to \mathcal{O}/\mathcal{I}_W \to 0. \tag{B.1}$$

$\mathcal{I}_W$ is the sheaf generated by the first derivatives of $W(x)$. This yields an exact sequence of rings of global sections

$$0 \to \Gamma(\mathbb{C}^k, \mathcal{I}_W) \to \mathbb{C}[x_1, \ldots, x_k] \to \Gamma(\mathbb{C}^k, \mathcal{O}/\mathcal{I}_W) \tag{B.2}$$

Since $W(x)$ is Morse, the derivatives $\partial_{x_1} W(x), \ldots, \partial_{x_k} W(x)$ can be regarded as local coordinates at each critical point p of $W(x)$. Thus, $\phi(x)$, which vanishes at p, can be written as $\sum_{a=1}^k g_a(x) \partial_{x_a} W(x)$ for some rational functions $g_a(x)$ which are regular in a neighborhood of p. Therefore, the image of $\phi(x)$ in $\Gamma(\mathbb{C}^k, \mathcal{O}/\mathcal{I}_W)$ vanishes. By the exactness of (B.2), $\phi(x)$ should come from $\Gamma(\mathbb{C}^k, \mathcal{I}_W)$. It remains to show $\Gamma(\mathbb{C}^k, \mathcal{I}_W) = I_W$, that is, any global section of $\mathcal{I}_W$ can be written as $\sum_{a=1}^k h_a(x) \partial_{x_a} W(x)$ for some polynomials $h_1(x), \ldots, h_k(x)$. For this, we consider another exact sequence of sheaves of $\mathcal{O}$-modules,

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus k} \to \mathcal{I}_W \to 0 \tag{B.3}$$

[5] We learned this proof from Kyoji Saito.

where the right map is defined by $(s_1(x), \ldots, s_k(x)) \mapsto \sum_{a=1}^{k} s_a(x)\partial_{x_a} W(x)$ and $\mathcal{K}$ is defined to be the kernel sheaf. This yields an exact sequence

$$\mathbb{C}[x_1, \ldots, x_k]^{\oplus k} \to \Gamma(\mathbb{C}^k, \mathcal{I}_W) \to \mathrm{H}^1(\mathbb{C}^k, \mathcal{K}) = 0. \tag{B.4}$$

This shows what we wanted.

C Convergence of integrals

Let us now discuss the convergence of the integral (5.1). The integrand is a pure phase. When $W(X)$ is a nontrivial function, the phase will oscillate greatly at infinity, which should guarantee the convergence. Here we analyze the issues of the convergence with more care. [6]

Let us consider in general an oscillatory integral

$$Z = \int_{\mathbb{R}^n} dx_1 \cdots dx_n \; \mathrm{e}^{iP(x_1,\ldots,x_n)} \tag{C.1}$$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and P is a real function. We assume that $|P'(x)|^2 := \sum_{j=1}^{n} |\partial_j P(x)|^2$ grows faster than a positive power of $|x|^2 = \sum_{j=1}^{n} |x_j|^2$ at infinity: there is some $\alpha > 0$ and $C > 0$

$$1 + |P'(x)|^2 \geq C(1 + |x|^2)^\alpha \qquad \text{for any } x. \tag{C.2}$$

The right hand side of (C.1) is only conditionally convergent. We would like to show that the absolutely convergent integral

$$Z_{\epsilon,f} = \int_{\mathbb{R}^n} dx_1 \cdots dx_n \; \mathrm{e}^{-\epsilon f(x_1,\ldots,x_n) + iP(x_1,\ldots,x_n)} \tag{C.3}$$

with a positive number ϵ and a positive function f which grows at least quadratically at infinity, has a limit when $\epsilon \to 0$ independent of f. We define this limit to be the left hand side of (C.1)

$$Z = \lim_{\epsilon \searrow +0} Z_{\epsilon,f} \tag{C.4}$$

independent of f. With this interpretation of the integral, the manipulation in Sec. 5 can be justified.

Let us introduce a differential operator

$$D := \frac{1}{1 + |P'(x)|^2} \left(1 - i \sum_{j=1}^{n} \partial_j P(x) \partial_j \right) \tag{C.5}$$

[6] The authors learned the treatment presented here from Alexander Getmanenko (a guidance including the reference [43]), Yoshitsugu Takei (useful comment) and Edward Witten (explicit instruction).

and its formal adjoint

$$L := \left(1 + i \sum_{j=1}^{n} \partial_j P(x) \partial_j\right) \frac{1}{1 + |P'(x)|^2} \times . \tag{C.6}$$

Using $D\,\mathrm{e}^{iP(x)} = \mathrm{e}^{iP(x)}$ we find

$$\begin{aligned} Z_{\epsilon.f} &= \int_{\mathbb{R}^n} \mathrm{d}^n x \; \mathrm{e}^{-\epsilon f(x)}\, D\,\mathrm{e}^{iP(x)} = \int_{\mathbb{R}^n} \mathrm{d}^n x\, L\left[\mathrm{e}^{-\epsilon f(x)}\right]\; \mathrm{e}^{iP(x)} \\ &\qquad \cdots \text{do it } N \text{ times} \cdots \\ &= \int_{\mathbb{R}^n} \mathrm{d}^n x\, L^N\left[\mathrm{e}^{-\epsilon f(x)}\right]\; \mathrm{e}^{iP(x)} \end{aligned} \tag{C.7}$$

The partial integration is valid as long as $\epsilon > 0$ due to the exponential decay. One can show that $L^N\left[\mathrm{e}^{-\epsilon f(x)}\right]$ decays as fast as $1/|P'(x)|^N$ for any $\epsilon \geq 0$ including $\epsilon = 0$. By the assumption (C.2), if we take N such that $N\alpha > n$, the right hand side of (C.7) is absolutely convergent for any $\epsilon \geq 0$. By the dominated convergence theorem, $Z_{\epsilon,f}$ has a limit as $\epsilon \searrow 0$ which is given by

$$\lim_{\epsilon \searrow 0} Z_{0,f} = \int_{\mathbb{R}^n} \mathrm{d}^n x\, L^N[1]\, \mathrm{e}^{iP(x)}, \tag{C.8}$$

for any N such that $N\alpha > n$. The result is obviously independent of f. This was what we wanted to demonstrate.

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